

Quantum Radiation from Quantum Gravitational Collapse

Tanmay Vachaspati and Dejan Stojkovic

CERCA, Department of Physics, Case Western Reserve University, Cleveland, OH 44106-7079

We study quantum radiation emitted during the collapse of a quantized, gravitating, spherical domain wall. The amount of radiation emitted during collapse now depends on the wavefunction of the collapsing wall and the background spacetime. If the wavefunction is initially in the form of a sharp wavepacket, the expectation value of the particle occupation number is determined as a function of time and frequency. The results are in good agreement with our earlier semiclassical analysis and show that the quantum radiation is non-thermal and evaporation accompanies gravitational collapse.

Quantum radiation from gravitationally collapsing matter has been the subject of a lot of interest [1, 2]. Virtually all of the work in this context has been done in the semiclassical approximation in which the collapsing matter is described in classical terms and only the radiation field is quantized. Such results, though very important, leave open the possibility that quantum collapse could qualitatively change the outcome. This would be relevant to both the end-point of Hawking radiation as well as to the dynamics of gravitational collapse close to the point of black hole formation. However, a full quantum treatment of gravitational collapse has been lacking so far.

In Ref. [3] a functional Schrodinger formalism approach to Hawking evaporation problem was developed, and the quantum evaporation from a classical collapsing background was studied. This formalism, being inherently quantum mechanical, is particularly convenient for treating the collapsing gravitational background in the context of quantum mechanics. In this paper we address the problem where the infalling matter and the radiation field are both treated in quantum theory.

Our approach follows Ref. [3] and uses the functional Schrodinger equation

$$H\Psi = i\frac{\partial\Psi}{\partial t} \quad (1)$$

where Ψ is the wavefunctional for a collapsing spherical domain wall and the excitation modes of the radiation field. We will find the Hamiltonian more explicitly below, after which we will solve the equation for the wavefunctional. The wavefunctional is used to find the excitation spectrum of the radiation field which leads to the flux of quantum radiation from the collapsing matter.

Our analysis goes a step beyond semiclassical analyses where the background spacetime is treated classically. However, the analysis does not include the backreaction of the radiation on the spacetime and collapsing matter.

To study a concrete realization of black hole formation we consider a spherical Nambu-Goto domain wall that is collapsing. To include the possibility of (spherically symmetric) radiation we consider a massless scalar field, Φ , that is coupled to the gravitational field but not directly

to the domain wall. The action for the system is

$$S = \int d^4x \sqrt{-g} \left[-\frac{\mathcal{R}}{16\pi G} + \frac{1}{2}(\partial_\mu \Phi)^2 \right] - \sigma \int d^3\xi \sqrt{-\gamma} \quad (2)$$

where the first term is the Einstein-Hilbert action for the gravitational field, the second is the scalar field action, the third is the domain wall action in terms of the wall world volume coordinates, ξ^a ($a = 0, 1, 2$), the wall tension, σ , and the induced world volume metric

$$\gamma_{ab} = g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \quad (3)$$

The coordinates $X^\mu(\xi^a)$ describe the location of the wall, and Roman indices go over internal domain wall world-volume coordinates ξ^a , while Greek indices go over space-time coordinates.

We will consider a minisuperspace version of the functional Schrodinger equation. As long as we keep all the degrees of freedom that are of interest to us, this is a useful truncation. With this in mind, we only consider spherical domain walls and assume spherical symmetry for all the fields. Then the wall is described by only the radial degree of freedom, $R(t)$. The metric is taken to be the solution of Einstein equations for a spherical domain wall. The metric is Schwarzschild outside the wall, as follows from spherical symmetry [4]

$$ds^2 = -\left(1 - \frac{R_S}{r}\right)dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1}dr^2 + r^2 d\Omega^2, \quad r > R(t) \quad (4)$$

where, $R_S = 2GM$ is the Schwarzschild radius in terms of the mass, M , of the wall, and $d\Omega^2$ is the usual angular line element. In the interior of the spherical domain wall, the line element is flat, as expected by Birkhoff's theorem,

$$ds^2 = -dT^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad r < R(t) \quad (5)$$

The interior time coordinate, T , is related to the observer time coordinate, t , via the proper time, τ , of the domain wall.

$$\frac{dT}{d\tau} = \left[1 + \left(\frac{dR}{d\tau} \right)^2 \right]^{1/2}, \quad \frac{dt}{d\tau} = \frac{1}{B} \left[B + \left(\frac{dR}{d\tau} \right)^2 \right]^{1/2} \quad (6)$$

where

$$B \equiv 1 - \frac{R_S}{R} \quad (7)$$

The ratio of these equations connects the interior and exterior time coordinates

$$\frac{dT}{dt} = \frac{(1 + R_\tau^2)^{1/2} B}{(B + R_\tau^2)^{1/2}} = \left[B - \frac{(1 - B)}{B} \dot{R}^2 \right]^{1/2} \quad (8)$$

where $R_\tau = dR/d\tau$ and $\dot{R} = dR/dt$. Integrating Eq. (8) still requires knowing $R(\tau)$ (or $R(t)$) as a function of τ (or t).

Since we are restricting our minisuperspace to fields with spherical symmetry, we need not include any other metric degrees of freedom. The scalar field can also be truncated to the spherically symmetric modes ($\Phi = \Phi(t, r)$).

It is important to point out that the Schwarzschild line element has a coordinate singularity at $r = R_S$. However, the restriction in Eq. (4) to $r > R(t)$ means that our coordinate system is well-defined everywhere as long as $R(t) > R_S$. In all that follows, we will only be considering the case $R(t) > R_S$. For example, we will calculate the quantum radiation from a collapsing wall with radius $R(t) > R_S$.

The energy function for the wall is [3]

$$H_{\text{wall}} = 4\pi\sigma B^{3/2} R^2 \left[\frac{1}{\sqrt{B^2 - \dot{R}^2}} - \frac{2\pi G\sigma R}{\sqrt{B^2 - (1 - B)\dot{R}^2}} \right] \quad (9)$$

where overdots denote derivatives with respect to t .

The canonical momentum is given by

$$\Pi \approx \frac{4\pi\mu R^2 \dot{R}}{\sqrt{B}\sqrt{B^2 - \dot{R}^2}} \quad (10)$$

where $\mu \equiv \sigma(1 - 2\pi G\sigma R_S)$. Then, in the regime $R \sim R_S$, the wall Hamiltonian is

$$H_{\text{wall}} \approx \frac{4\pi\mu B^{3/2} R^2}{\sqrt{B^2 - \dot{R}^2}} \quad (11)$$

$$= [(B\Pi)^2 + B(4\pi\mu R^2)^2]^{1/2} \quad (12)$$

and has the form of the energy of a relativistic particle, $\sqrt{p^2 + m^2}$, with a position dependent mass. In the limit $B \sim 0$, the mass term can be neglected – the wall is ultra-relativistic – and hence

$$H_{\text{wall}} \approx -B\Pi \quad (13)$$

where we have chosen the negative sign appropriate for describing a collapsing wall.

The scalar field, Φ , is decomposed into a complete set of basis functions denoted by $\{f_k(r)\}$

$$\Phi = \sum_k a_k(t) f_k(r) \quad (14)$$

The exact form of the functions $f_k(r)$ will not be important for us. We will be interested in the wavefunction for the mode coefficients $\{a_k\}$.

The Hamiltonian for the scalar field modes is found by inserting the scalar field mode decomposition and the background metric into the action

$$S_\Phi = \int d^4x \sqrt{-g} \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \quad (15)$$

The Hamiltonian for the scalar field modes takes the form of coupled simple harmonic oscillators with R -dependent mass and couplings due to the non-trivial metric. In the regime $R \sim R_S$, for a normal mode denoted by b , the Hamiltonian is [3]

$$H_b = \left(1 - \frac{R_S}{R}\right) \frac{\pi^2}{2m} + \frac{K}{2} b^2 \quad (16)$$

where π is the momentum conjugate to b , m and K are constants whose precise values are not important for us.

Hence the total quantum Hamiltonian for the wall and a normal mode of the scalar field is

$$H = H_{\text{wall}} + H_b = -B\Pi + B \frac{\pi^2}{2m} + \frac{K}{2} b^2 \quad (17)$$

where $\Pi = -i\partial/\partial R$, $\pi = -i\partial/\partial b$. The wavefunction is now a function of b , R and t : $\Psi = \Psi(b, R, t)$.

The differential form of the Schrodinger equation is

$$+iB \frac{\partial \Psi}{\partial R} - \frac{B}{2m} \frac{\partial^2 \Psi}{\partial b^2} + \frac{K}{2} b^2 \Psi = i \frac{\partial \Psi}{\partial t} \quad (18)$$

We look for stationary solutions

$$\Psi(b, R, t) = e^{-iEt} \psi(b, R) \quad (19)$$

Then the time-independent Schrodinger equation is

$$- \frac{1}{2m} \frac{\partial^2 \psi}{\partial b^2} + \frac{m}{2} \omega^2 b^2 \psi - \epsilon \psi = -i \frac{\partial \psi}{\partial R} \quad (20)$$

where $\omega^2 = k/(mB)$ and $\epsilon = E/B$. To solve Eq. (20) we use the ansatz

$$\psi(b, R) = e^{-i \int^R \epsilon(R') dR'} \phi(b, R) = e^{-iEu} \phi(b, R) \quad (21)$$

where

$$u = \int^R \frac{dR'}{B(R')} = R + R_S \ln \left| \frac{R}{R_S} - 1 \right| \quad (22)$$

This leads to the equation for $\phi(b, \eta)$ where $\eta \equiv R_S - R$

$$- \frac{1}{2m} \frac{\partial^2 \phi}{\partial b^2} + \frac{m\omega^2}{2} b^2 \phi = i \frac{\partial \phi}{\partial \eta} \quad (23)$$

As discussed in Ref. [5], this has the implicit solution

$$\phi(b, \eta) = e^{i\alpha(\eta)} \left(\frac{m}{\pi\rho^2} \right)^{1/4} \exp \left[\frac{im}{2} \left(\frac{\rho_\eta}{\rho} + \frac{i}{\rho^2} \right) b^2 \right] \quad (24)$$

where ρ_η is the derivative of the function $\rho(\eta)$ with respect to η , and the defining equation for ρ is

$$\rho_{\eta\eta} + \omega^2(\eta)\rho = \frac{1}{\rho^3} \quad (25)$$

where

$$\omega^2(\eta) = -\frac{kR}{m\eta} \approx -\frac{kR_S}{m\eta} \quad (26)$$

The initial conditions for ρ are taken at some large negative value of η (*i.e.* large value of R) denoted by η_i , so that

$$\rho(\eta_i) = \frac{1}{\sqrt{\omega(\eta_i)}}, \quad \rho_\eta(\eta_i) = 0 \quad (27)$$

The phase α is defined by

$$\alpha(\eta) = -\frac{1}{2} \int^\eta \frac{d\eta'}{\rho^2(\eta')} \quad (28)$$

Then Eqs. (19) and (21) give

$$\Psi(b, R, t) = e^{-iE(u+t)} \phi(b, \eta) \quad (29)$$

with ϕ given in Eq. (24) and, as above, $\eta = R_S - R$.

Next we note that E only enters the solution in the first exponential factor. So we can superpose stationary solutions to construct wavepackets. For example,

$$\Psi(b, R, t) = \frac{1}{(\pi\sigma^2)^{1/4}} e^{-(u+t)^2/2\sigma^2} \phi(b, \eta) \quad (30)$$

where σ is the width of the wavepacket, is a solution to the time dependent Schrodinger problem in Eq. (18). With the chosen initial conditions for ρ at η_i , this solution describes a wavepacket for the collapsing wall and quantum fields in their ground state at η_i .

Note that we have normalized the wavepacket with unit measure in the u coordinate since the Schrodinger evolution preserves this normalization. Also, u can be expressed in terms of η through Eq. (22).

The solution (30) describes quantum radiation from a quantized collapsing shell of matter. The shell is represented by a Gaussian wave packet that is moving toward the horizon R_S which corresponds to $u \rightarrow -\infty$. Radiation from the shell, represented by $\phi(b, \eta)$ depends on the position of the shell. We want to find the main features of this radiation.

It is worth noting that

$$\Psi(b, R, t) = f(u+t) \phi(b, \eta) \quad (31)$$

is a solution for any function f , provided $\phi(b, \eta)$ satisfies Eq. (23). This solution shows that wavepackets in u do not spread in time at late times. Since $dR = Bdu$ from Eq. (22) and B gets smaller as the wall collapses, the wavepacket gets more sharply peaked in R with time.

In the semiclassical analysis where R is treated classically, the wavefunction for the mode amplitudes is decomposed into suitably chosen basis wavefunctions (discussed below). If a complete basis is denoted by $\{\phi_n\}$, then the expectation of the occupation number is

$$N = \sum_n n |\langle \phi_n | \Psi \rangle|^2 \quad (32)$$

Similar considerations hold for the present case where we are treating R in quantum theory. The only difference is that the value of N depends on what value R takes. Hence N is given by an R - and t - dependent probability distribution, and to find the expectation value of N at any time, we also need to integrate over R . Therefore,

$$\langle N(t) \rangle = \int du \langle N(R, t) \rangle = \int du \sum_n n |\langle \phi_n | \Psi \rangle|^2 \quad (33)$$

Here we have been careful to integrate over du with unit measure which is equivalent to integrating over R but with measure $1/B$.

To proceed further, we need to specify our basis wavefunctions, ϕ_n . These are chosen to be simple harmonic oscillator basis states at a frequency $\bar{\omega}$

$$\phi_n(b) = \left(\frac{m\bar{\omega}}{\pi}\right)^{1/4} \frac{e^{-m\bar{\omega}b^2/2}}{\sqrt{2^n n!}} H_n(\sqrt{m\bar{\omega}}b) \quad (34)$$

where H_n are Hermite polynomials. Then [3]

$$\langle \phi_n | \Psi \rangle = f(u+t) \frac{(-1)^{n/2} e^{-i\alpha}}{(\bar{\omega}\rho^2)^{1/4}} \sqrt{\frac{2}{P}} \left(1 - \frac{2}{P}\right)^{n/2} \frac{(n-1)!!}{\sqrt{n!}} \quad (35)$$

where $f(u+t)$ can be chosen to be a Gaussian function as in Eq. (30) and

$$P \equiv 1 - \frac{i}{\bar{\omega}} \left(\frac{\rho_\eta}{\rho} + \frac{i}{\rho^2} \right) \quad (36)$$

The sum over n in Eq. (33) can be done explicitly [3] and the occupation number is

$$\langle N(t) \rangle = \int du |f(u+t)|^2 \frac{\bar{\omega}\rho^2}{\sqrt{2}} \left[\left(1 - \frac{1}{\bar{\omega}\rho^2}\right)^2 + \left(\frac{\rho_\eta}{\bar{\omega}\rho}\right)^2 \right] \quad (37)$$

Note that the integrand contains ρ which is a function of $\eta = R_S - R$ (Eq. (25)), which in turn is a function of u (Eq. (22)). The frequency ω in Eq. (26) depends on R and hence is a quantum variable. The frequency $\bar{\omega}$ at which N is evaluated is chosen to be the expectation value of ω : $\bar{\omega} = \langle \omega \rangle$.

To evaluate $\langle N \rangle$ we now need to solve Eq. (25) and then insert the solution in the integrand in Eq. (37) together with the Gaussian form of $f(u+t)$. Finally, we need to do the u integral in Eq. (37). If $f(u+t)$ is sharply peaked, only a small window of u , and an even smaller window of R (Eq. (22)), contributes to the integral.

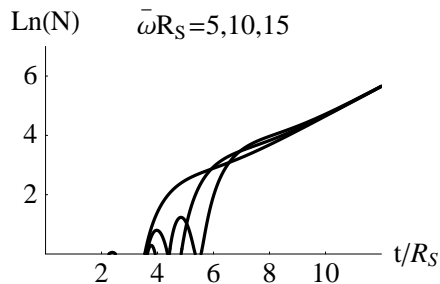


FIG. 1: $\ln(N)$ versus t/R_S for various fixed values of $\bar{\omega}R_S$ and for the wavepacket width $\sigma = 0.1R_S$. Here, $\bar{\omega}$ is the expectation value of the frequency calculated for the state (30). The linear growth shows that the occupation number increases exponentially at late times.

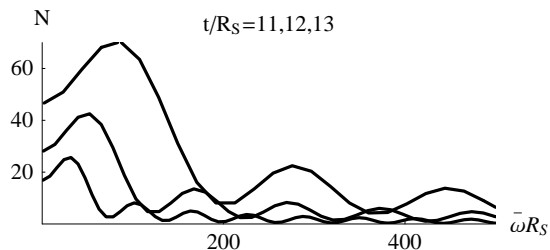


FIG. 2: N versus $\bar{\omega}R_S$ for various fixed values of t/R_S for $\sigma = 0.1R_S$. The oscillations depend on the choice of width of the wavepacket σ . Here, $\bar{\omega}$ is the expectation value of the frequency calculated for the state (30). The occupation number at any frequency grows as t/R_S increases.

We have calculated the occupation number numerically for the case of a wavepacket with $\sigma = 0.1R_S$ and we show the plot of $\langle N(t) \rangle$ versus t in Fig. 1. These calculations, when done for a range of $\bar{\omega}$, also yield the spectrum of occupation numbers. In Fig. 2 we show the spectrum for a few different times. This result has the same general features as in the semiclassical analysis [3]. The main effect of the quantized background is that the spectrum becomes more non-thermal though this depends on the width of the wavepacket. In the case of a very sharp wavepacket ($\sigma \rightarrow 0$), the spectrum becomes identical to the semiclassical spectrum.

In summary, we have studied quantum radiation from the changing spacetime metric due to a collapsing spherical domain wall. The main improvement over earlier calculations is that the domain wall, and hence the metric, are treated within quantum theory. The end results for the growth of particle occupation number and spectrum are very similar to that found in semiclassical analysis [3] and, in fact, become identical to the semiclassical result as the width of the wavepacket decreases.

Our analysis implies that quantum radiation is emitted continuously during quantum gravitational collapse, and so evaporation and collapse are concomitant. As seen in Fig. 1, the occupation number at any frequency grows

with time, and as in Fig. 2, the spectrum has non-thermal features. In an earlier paper [3], we have approximately fitted a thermal distribution to the spectrum in an intermediate range of frequencies, and found a radiation roughly consistent with the Hawking temperature. As the collapse proceeds, the window of frequencies in which the radiation is thermal grows, but only in the $t \rightarrow \infty$ limit, does the spectrum become thermal in an infinite range of frequencies. The thermal spectrum, however, may never be achieved in a physical setting precisely because it is only realized in the $t \rightarrow \infty$ limit.

We should note that our analysis has ignored radiation backreaction on the collapse process, since we take R_S to be constant. A full study of backreaction must involve time dependence of R_S . However, in Ref. [3] we have argued that, since steady evaporation accompanies gravitational collapse (*i.e.* for $R(t) > R_S$), all the energy in the collapsing object is radiated away in a finite time and before a black hole can form. Since our results for the radiation rate for the quantum gravitational collapse are very similar to those in the semiclassical treatment, we have nothing more to add to the discussion but we summarize the argument here for completeness.

Due to the backreaction of evaporation, the Schwarzschild radius, R_S , must decrease with time. On the other hand, as the collapse proceeds, the gravitational time delay from the surface of the shell to the asymptotic observer becomes larger and would become infinite if the shell were to ever cross its (instantaneous) Schwarzschild radius. So no signal from the collapsing shell can arrive to the asymptotic observer so as to convey that a black hole has formed and neither can any object be seen to disappear into a black hole at any finite time. But the asymptotic observer does receive the full energy of the infalling matter in a finite time, given by the evaporation time. The only logical conclusion is that the collapsing object evaporates before a black hole can form. To confirm this argument in detail requires a backreaction calculation that we hope to return to in future.

This work was supported by the U.S. Department of Energy and NASA at Case Western Reserve University.

-
- [1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975) [Erratum-ibid. **46**, 206 (1976)].
 - [2] “Quantum Fields in Curved Space”, N.D. Birrell and P.C.W. Davies, Cambridge University Press (1982).
 - [3] T. Vachaspati, D. Stojkovic and L.M. Krauss, gr-qc/0609024.
 - [4] J. Ipser and P. Sikivie, Phys. Rev. D **30**, 712 (1984).
 - [5] C. M. A. Dantas, I. A. Pedrosa and B. Baseia, Phys. Rev. A **45**, 1320 (1992).
Hole,”